AIAA Journal
Vol. 30, No. 10, October 1992

Analysis of Forced Bilinear Oscillators and the Application to Cracked Beam Dynamics

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A new closed-form solution based on the use of two square wave functions to model stiffness change is proposed for bilinear oscillators under low-frequency excitation. The solution procedure is extended to cases with bilinear forcing function, which usually occur in the dynamics of damaged structures. The proposed solution provides the spectral pattern and the magnitude of each harmonic component for a damaged rectangular beam according to the size and location of the crack. The predicted spectral pattern and the magnitude of each harmonic component for a simply supported beam containing a fatigue crack are validated by numerical simulation; experimental work has not been done. The results show that the proposed solution may be used to predict efficiently spectral patterns of damaged structures, while avoiding lengthy computation. Such a procedure may be developed into an attractive on-line diagnostic tool.

Nomenclature

A	= beam cross section
a	= crack depth
b	= half-breadth of rectangular beam
d	= half-depth of rectangular beam
E	= Young's modulus of elasticity
$F(t)$	= forcing function
$f_k(u)$	= piecewise linear restoring forcing function
I	= cross-sectional area moment of inertia
k_1	= stiffness constant as $u>0$
k_2	= stiffness constant as $u<0$
m	= stress magnification factor
m_0	= mass of oscillator
T_0	= period of ϕ_1 and ϕ_2
T_1	= semiperiod of ϕ_1 and ϕ_2 as $u>0$
T_2	= semiperiod of ϕ_1 and ϕ_2 as $u<0$
t_0	= time at transition point
u	= displacement of oscillator
x_c	= crack position
α	= stress decay constant
α_0	= period angle for periodic square wave functions
β	= neutral surface shift constant
β_0	= phase angle for $u<0$
γ	= contact parameter
ρ	= density
ϕ_1	= periodic square wave function for $u>0$ domain
ϕ_2	= periodic square wave function for $u<0$ domain
ψ_i^c	= mode shapes of cracked beams
ψ_i^{nc}	= mode shapes of uncracked beams
Ω	= frequency of harmonic excitation
ω_0	= bilinear frequency
ω_1	= governing frequency as $u>0$
ω_2	= governing frequency as $u<0$

Introduction

THE use of simplified models to simulate the nonlinear behavior of a real system has been a very challenging task in mechanics. Because of their simplicity, oscillators with bilinear or piecewise stiffness characteristics (i.e., with a bilinear or piecewise linear restoring forcing function) have often been used to model the behavior of mechanical systems with nonlinear stiffness. For example, Moon and Shaw[1] modeled cantilever beams with nonlinear boundary conditions by oscillators with bilinear stiffness. Ehrich[2] modeled the bearing suspension of high-speed rotors by oscillators with piecewise stiffness. Thompson and Elvey[3] used a bilinear restoring force to model the nonlinearity of a wave-driven off-shore tower. Natsiavas and Babcock[4] modeled unanchored fluid-filled tanks subjected to ground excitation by a forced bilinear oscillator. Neilson and Barr[5] used a bilinear restoring force to model the nonlinear elastic supports of a rigid rotor. Tung and Shaw[6] used a forced bilinear oscillator with a stop to simulate the behavior of print hammers.

Recently the use of bilinear oscillators has been extended to model the dynamic behavior of structures with a fatigue crack.[7-12] A so-called breathing crack was introduced based on the bilinear stiffness concept, which assumes that the crack opens (with lower stiffness) or closes (with higher stiffness) depending on the direction of the vibration. For example, Ibrahim et al.[7] used the bondgraph technique (a numerical tool using lumped parameter elements) to examine the frequency response function of a cracked cantilever beam. Zastrau[8] used a finite element method to study the time history and spectrum of a simply supported beam with multiple cracks. Collins et al.[9] used direct numerical integration to investigate longitudinal vibrations of a cracked prismatic bar. Qian et al.[10] applied time series analysis to identify the modal parameters of cracked beams. Shen and Chu[11,12] proposed a modified cracked beam theory and simulated numerically the dynamic response of simply supported beams with a fatigue crack. Shen and Chu found that the spectral patterns for damaged and undamaged beams were very different. If the change in the spectral pattern due to damage can be predicted efficiently, an effective on-line damage detection technique can be developed for structures with fatigue cracks.

Since all numerical simulation techniques require tremendous computational time during the analysis, we consider a closed-form solution a better alternative for determining the characteristics of the spectral pattern for an on-line damage detection/identification system. In addition, the closed-form solution provides physical interpretation of spectral information. However, previous research on the dynamics of bilinear or piecewise linear systems has focused mainly on the steady-state superharmonic or subharmonic resonances, or bifurcation and limit cycles of nonlinear motions.[13-21] These studies cannot be used to predict the spectrum pattern or to estimate

Received July 15, 1991; revision received April 6, 1992; accepted for publication April 6, 1992. Copyright © 1992 by Y. C. Chu and M.-H. H. Shen. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission.

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the magnitude of each harmonic component. The objective of the present paper is to develop a closed-form solution for bilinear oscillators under low-frequency harmonic excitation. A new modeling technique using two periodic square wave functions to model the change of stiffness during vibration is proposed in this study. The advantage of using this stiffness modeling technique is that both the homogeneous solution and the particular solution can be expressed in terms of these two periodic square wave functions. By expanding these two periodic square wave functions in Fourier series and introducing transition conditions to define the continuity of both displacement and velocity between different stiffness domains, a closed-form solution can be obtained in a Fourier series form. Each term in this series solution naturally corresponds to a harmonic component in the spectrum of bilinear oscillators; hence, the solution provides both the overall pattern and each individual harmonic component of the spectrum.

The solution for bilinear oscillators under constant amplitude harmonic excitation is developed first and is valid for any mechanical system that can be modeled by a bilinear oscillator. However, for cracked beam dynamics, we have shown[11] that each mode of cracked beam vibrations should be modeled as a bilinear oscillator excited by a bilinear forcing function, instead of as a constant amplitude excitation, as was suggested in Ref. 8. Under this assumption, the amplitude of the harmonic forcing function changes, depending on the change of the stiffness constants. Thus, the aforementioned solution procedure is further extended to incorporate the bilinear forcing function, and a new general solution for forced cracked beam vibrations is developed.

To verify the solution obtained by the proposed modeling technique, results predicted by the present solution are compared with spectra obtained from numerical simulation for bilinear oscillators under constant-amplitude excitation and for vibrations of cracked beams. It is shown that in both cases the present solutions successfully predict the overall pattern and each harmonic component, compared with the simulations. Experimental verification is not performed in this study and will be undertaken in the near future.

Since the bilinear behavior of a cracked structure is heavily dependent on its damage properties, the fact that harmonic components for cracked and uncracked beams are different clearly indicates crack existence. Using the present solution, one can predict the spectrum of vibrations of cracked beams. Thus, we believe that a diagnostic procedure to identify the damage can be developed using the spectral pattern obtained from vibration tests.

Modeling of Bilinear Stiffness and Solutions for Free Vibrations

A harmonic oscillator with bilinear stiffness and mass m_0, as shown in Fig. 1, is considered here. The equation of motion for this bilinear oscillator is

$$\begin{cases} \ddot{u} + \omega_1^2 u = F(t) & \text{if} \quad u > 0 \\ \ddot{u} + \omega_2^2 u = F(t) & \text{if} \quad u < 0 \end{cases} \tag{1}$$

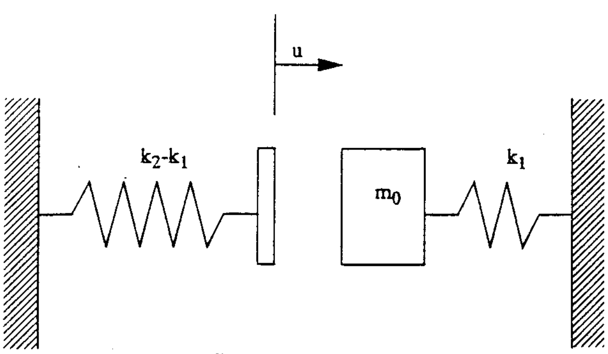

Fig. 1 Configuration of a bilinear oscillator.

where $F(t)$ is the external excitation, u is the displacement of the oscillator, overdots indicate $\partial/\partial t$, and

$$\omega_1^2 = k_1/m_0, \qquad \omega_2^2 = k_2/m_0 \tag{2}$$

To obtain a closed-form solution, we introduce a special technique using two square wave functions to model the change of stiffness during vibration. In this case, Eq. (2) can be expressed in terms of two square wave functions $\phi_1(t)$ and $\phi_2(t)$ as

$$\ddot{u} + [\omega_1^2\phi_1(t) + \omega_2^2\phi_2(t)]u = 0 \tag{3}$$

where the square wave functions $\phi_1(t)$ and $\phi_2(t)$ are defined by

$$\phi_1(t) = \begin{cases} 1 & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u < 0 \end{cases} \tag{4}$$

and

$$\phi_2(t) = 1 - \phi_1(t) \tag{5}$$

The derivatives of $\phi_1(t)$ and $\phi_2(t)$ with respect to time are defined by

$$\begin{cases} \left(\dfrac{d}{dt}\right)^n \phi_i = 0 & \text{if} \quad u \neq 0 \\ \left(\dfrac{d}{dt}\right)^n \phi_i = \text{finite} & \text{if} \quad u = 0 \end{cases} \tag{6}$$

Such properties allow the solution of the homogeneous equation to be expressed in terms of $\phi_1(t)$ and $\phi_2(t)$ as

$$u_h(t) = A_1\phi_1 \sin \omega_1 t + A_2\phi_2 \sin \omega_2(t - t_0) + B_1\phi_1 \cos \omega_1 t + B_2\phi_2 \cos \omega_2(t - t_0) \tag{7}$$

where t_0 is the time when the transition occurs; A_1, A_2, B_1, and B_2 are integration constants to be determined; A_1 and B_1 are determined by the initial conditions; and A_2 and B_2 are determined by the transition conditions at the transition point if the initial conditions are within the $u > 0$ domain, and vice versa. The transition conditions require the continuity of both displacement and velocity at the stiffness transition point (for the present case, it is the point where $u = 0$), which are defined as

$$\dot{u}(t_{0+}) = \dot{u}(t_{0-}) \qquad \text{at} \qquad u = 0 \tag{8}$$

where $\dot{u}(t_{0+})$ is the velocity of the oscillator when it moves from the positive domain to zero, and $\dot{u}(t_{0-})$ is the velocity when it moves from zero to the negative domain.

Free vibration with initial conditions $u(0) = 1$ and $\dot{u}(0) = 0$ is considered here. A_1 and B_1 due to these initial conditions are given by

$$A_1 = 0, \qquad B_1 = u(0) = 1 \tag{9}$$

A_2 and B_2 due to the transition conditions are given by

$$A_2 = -(\omega_1/\omega_2), \qquad B_2 = 0 \tag{10}$$

By substituting all four of these constants into Eq. (7), the solution to the free vibration becomes

$$u(t) = \phi_1(t) \cos \omega_1 t + (\omega_1/\omega_2)\phi_2(t) \cos(\omega_2 t + \beta_0) \tag{11}$$

where β_0 is given by

$$\beta_0 = \pi/2[1 - (\omega_2/\omega_1)] \tag{12}$$

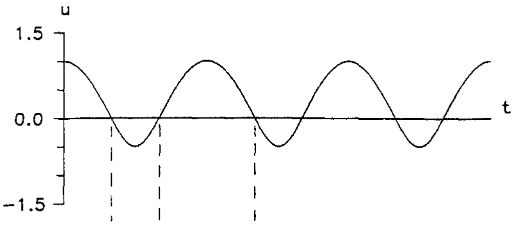

Fig. 2a Free vibration of a bilinear oscillator.

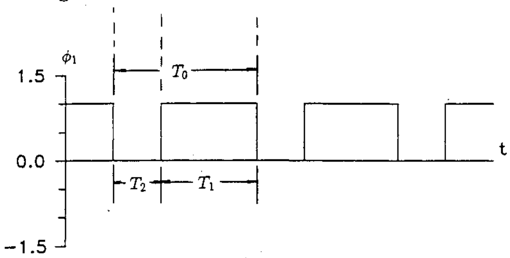

Fig. 2b Periodic square wave function ϕ_1.

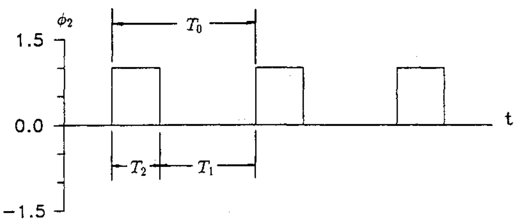

Fig. 2c Periodic square wave function ϕ_2.

In this case the motion of the oscillator is periodic as shown in Fig. 2a. Thus, both $\phi_1(t)$ and $\phi_2(t)$ are periodic functions with one semiperiod $T_1 = \pi/\omega_1$ and another semiperiod $T_2 = \pi/\omega_2$ (see Figs. 2b and 2c). The period of these two periodic functions T_0 is given by

$$T_0 = T_1 + T_2 = 2\pi/\omega_0 \tag{13}$$

where the bilinear frequency ω_0 is defined as

$$\omega_0 = (2\omega_1\omega_2)/(\omega_1 + \omega_2) \tag{14}$$

Since both $\phi_1(t)$ and $\phi_2(t)$ are periodic functions, they can be expressed as Fourier series:

$$\frac{a_2}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{15}$$

where

$$\begin{cases} a_n = \dfrac{2}{T_0}\displaystyle\int_0^{T_0} \phi_i(t) \cos n\omega_0 t \, \mathrm{d}t; \quad n = 0, \dots, \infty \\ b_n = \dfrac{2}{T_0}\displaystyle\int_0^{T_0} \phi_i(t) \sin n\omega_0 t \, \mathrm{d}t; \quad n = 1, \dots, \infty \end{cases} \tag{16}$$

The resulting series forms of $\phi_1(t)$ and $\phi_2(t)$ are given by

$$\phi_1(t) = \frac{\omega_0}{2\omega_1} + \frac{2}{\pi} + \left(\sum_{k=1}^{\infty} \frac{\sin k\alpha_0}{k} \cos k\omega_0 t\right) \tag{17}$$

and

$$\phi_2(t) = \left(1 - \frac{\omega_0}{2\omega_1}\right) - \frac{2}{\pi} + \left(\sum_{k=1}^{\infty} \frac{\sin k\alpha_0}{k} \cos k\omega_0 t\right) \tag{18}$$

where α_0 is given by

$$\alpha_0 = \pi\omega_0/2\omega_1 \tag{19}$$

By substituting Eqs. (17) and (18) into Eq. (11), the solution of Eq. (11) can be rewritten by

$$\begin{aligned} u(t) = {} & \frac{\omega_0}{2\omega_1} \cos \omega_1 t + \left(1 - \frac{\omega_0}{2\omega_1}\right)\frac{\omega_1}{\omega_2} \cos(\omega_2 t + \beta_0) \\ & + \frac{1}{\pi}\left\{\sum_{k=1}^{\infty} \frac{\sin k\alpha_0}{k} \cos(\omega_1 \pm k\omega_0)t \right. \\ & \left. - \sum_{k=1}^{\infty} \frac{\sin k\alpha_0}{k} \cos[(\omega_2 \pm k\omega_0)t + \beta_0]\right\} \end{aligned} \tag{20}$$

If $\omega_1 = \omega_2$ or $\omega_0 = \omega_1 = \omega_2$, a limiting case of Eq. (20), the values of the parameters α_0 and β_0 simplify to

$$\alpha_0 = \pi/2, \qquad \beta_0 = 0 \tag{21}$$

The solution of the limiting case reduces to

$$u(t) = \cos \omega_1 t \tag{22}$$

which is a simple oscillator.

Solutions for Low-Frequency Forced Vibrations

For low-frequency forced vibrations, the external excitation is given as

$$F(t) = p \sin \Omega t \tag{23}$$

where P is the magnitude of excitation, and the forcing frequency Ω is assumed to be much smaller than both natural frequencies ω_1 and ω_2 [i.e., $\Omega \ll \min(\omega_1, \omega_2)$]. The particular solution of Eq. (1) for different stiffness domains can be obtained by simply introducing a harmonic function with different amplitudes in different stiffness domains:

$$\begin{cases} u_p(t) = \dfrac{P}{\omega_1^2 - \Omega^2} \sin \Omega t \quad \text{if} \quad u > 0 \\ u_p(t) = \dfrac{P}{\omega_2^2 - \Omega^2} \sin \Omega t \quad \text{if} \quad u < 0 \end{cases} \tag{24}$$

This can also be written in terms of the square wave functions as

$$u_p(t) = \phi_1(t)\frac{P}{\omega_1^2 - \Omega^2} \sin \Omega t + \phi_2(t)\frac{P}{\omega_2^2 - \Omega^2} \sin \Omega t \tag{25}$$

The general solution for the forced vibration can be obtained by combining the particular solution [Eq. (25)] and the homogeneous solution [Eq. (7)], which yields

$$\begin{aligned} u(t) = {} & A_1\phi_1 \sin \omega_1 t + A_2\phi_2 \sin \omega_2(t - t_0) + B_1\phi_1 \cos \omega_1 t \\ & + B_2\phi_2 \cos \omega_2(t - t_0) + \phi_1 \frac{P}{\omega_1^2 - \Omega^2} \sin \Omega t \\ & - \phi_2 \frac{P}{\omega_2^2 - \Omega^2} \sin \Omega(t - t_0) \end{aligned} \tag{26}$$

For systems with homogeneous initial conditions [i.e., $u(0) = \dot{u}(0) = 0$], the constants are given by

$$\begin{aligned} \beta_0 &= -(\pi\omega_2/\Omega) \\ B_1 &= B_2 = 0 \\ A_1 &= -(\Omega/\omega_1)(P/\omega_1^2 - \Omega^2) \\ A_2 &= (\Omega/\omega_2)(P/\omega_2^2 - \Omega^2) \end{aligned} \tag{27}$$

By substituting the constants in Eqs. (27) into Eq. (26), the solution for forced vibration of this bilinear system with homogeneous initial conditions becomes

$$u(t) = \phi_1(t)\left[\frac{P}{\omega_1^2 - \Omega^2}\sin\Omega t + A_1 \sin\omega_1 t\right] + \phi_2(t)\left[\frac{P}{\omega_2^2 - \Omega^2}\sin\Omega t + A_2 \sin(\omega_2 t + \beta_0)\right] \quad (28)$$

Since the present study mainly focuses on low-frequency forced vibrations, only the case where the frequency of the harmonic excitation is much smaller than the lower natural frequency of the bilinear system is considered. In this case the forced response terms $P/(\omega_1^2 - \Omega_2)\sin\Omega t$ and $P/(\omega_2^2 - \Omega^2)\sin\Omega t$ dominate the motion of the oscillator (both A_1 and A_2 are very small). Thus, the square wave functions $\phi_1(t)$ and $\phi_2(t)$ are forced to synchronize with the harmonic excitation since their period is determined by the motion of the oscillator. This synchronization allows us to make an approximation by assuming that the square wave functions $\phi_1(t)$ and $\phi_2(t)$ are periodic and have the same frequency as the harmonic excitation, as shown in Fig. 3. Using the expansion procedure described in the previous section, $\phi_1(t)$ and $\phi_2(t)$ can be expanded in Fourier series form as

$$\phi_1(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1,3,5}^{\infty}\frac{\sin k\Omega t}{k} \quad (29)$$

and

$$\phi_2(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1,3,5}^{\infty}\frac{\sin k\Omega t}{k} \quad (30)$$

which leads to

$$u(t) = u_f(t) + u_s(t) \quad (31)$$

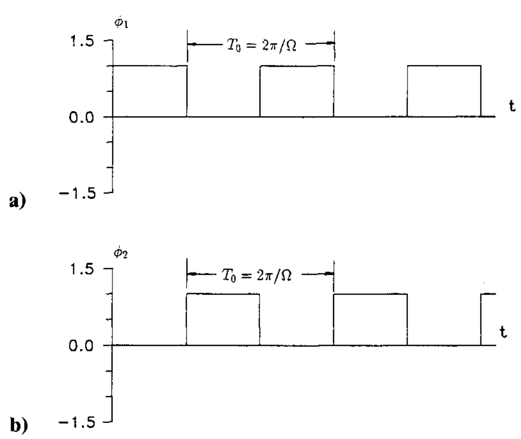

Fig. 3 Periodic square wave functions for the forced vibrations of a bilinear oscillator: a) ϕ_1; b) ϕ_2.

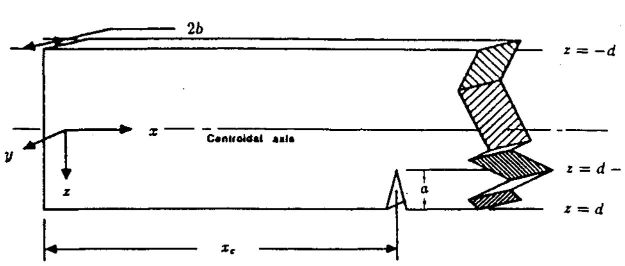

Fig. 4 Configuration of the cracked beam.

where $u_f(t)$ is the forced response of system, and $u_s(t)$ is the system resonance. These two solutions are given by

$$u_f(t) = \frac{1}{2}\left(\frac{P}{\omega_1^2 - \Omega^2} + \frac{P}{\omega_2^2 - \Omega^2}\right)\sin\Omega t + \left(\frac{P}{\omega_1^2 - \Omega^2} - \frac{P}{\omega_2^2 - \Omega^2}\right)\left\{\frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\Omega t}{(2k+1)(2k-1)}\right\} \quad (32)$$

and

$$u_s(t) = \frac{1}{2}[A_1 \sin\omega_1 t + A_2 \sin(\omega_2 t + \beta_0)] + \frac{A_1}{\pi}\sum_{k=1,3,5}^{\infty}\frac{\cos(\omega_1 - k\Omega)t - \cos(\omega_1 + k\Omega)t}{k} - \frac{A_2}{\pi} \times \sum_{k=1,3,5}^{\infty}\frac{\cos[(\omega_2 - k\Omega)t + \beta_0] - \cos[(\omega_2 + k\Omega)t + \beta_0]}{k} \quad (33)$$

The preceding solution, Eqs. (32) and (33), clearly provides a specific spectral pattern for both the system resonance and the forced response for low-frequency forced vibrations. Since each term in the preceding solution represents a harmonic component in the spectrum, it is possible to determine the magnitude of each harmonic component from this closed-form solution. Again, as a limiting case when $\omega_1 = \omega_2$, Eq. (31) reduces to

$$u(t) = \frac{P}{\omega_1^2 - \Omega^2}\sin\Omega t + A_1 \sin\omega_1 t \quad (34)$$

which is the solution for a forced simple harmonic oscillator.

Applications to Cracked Beam Dynamics

We have shown[11] that the governing equation for beams with a fatigue crack obtained from the Galerkin procedure has not only bilinear stiffness but also bilinear forcing functions. In this case the magnitude of the external excitation for each mode of vibration indeed changes with the stiffness constant. To analyze the vibrations of cracked beams, it is necessary to extend the stiffness modeling technique to incorporate the bilinear forcing function. We consider a beam of rectangular cross section of depth $2d$ and width $2b$, with an edge crack of size a, located at $x = x_c$, as shown in Fig. 4. The equation of motion for the ith mode of vibrations is given by[11]

$$m_{0i}\ddot{u}_i + f_k(u_i) = F_{1i}\sin\Omega t \quad (35)$$

where the restoring force $f_k(u_i)$ is given by

$$f_k(u_i) = \begin{cases} k_{1i}u_i & \text{if} \quad u_i > 0 \\ k_{2i}u_i & \text{if} \quad u_i < 0 \end{cases} \quad (36)$$

The mass m_{0i}, the magnitude of the harmonic excitation F_{1i}, and the stiffness constants k_{1i} and k_{2i} are given by

$$m_{0i} = \frac{\rho A}{E}\int_0^l \psi_i\psi_i \, dx \quad (37)$$

$$F_{1i} = \gamma\int_0^l \xi\psi_i^c \, dx + (1-\gamma)\int_0^l \xi\psi_i^{nc} \, dx \quad (38)$$

$$k_{1i} = \int_0^l \psi_i^c[r(x)\psi_i^{c\,''''} + g(x)\psi_i^{c\,'''} + h(x)\psi_i^{c\,''}] \, dx \quad (39)$$

and

$$k_{2i} = \int_0^l \psi_i^{nc}[r(x)\psi_i^{nc\,''''} + g(x)\psi_i^{nc\,'''} + h(x)\psi_i^{nc\,''}] \, dx \quad (40)$$

The function ψ_i^c is the ith mode shape of the cracked beam, ψ_i^{nc} is the ith mode shape of the uncracked beam, the functions ψ_i

are mode shapes of either the uncracked or the cracked beam, $\xi(x)$ is the loading distribution of the harmonic excitation, and the relation between the contact parameter γ and the displacement u_i is given by

$$\gamma = \begin{cases} 1 & \text{as} \quad u_i > 0 \\ 0 & \text{as} \quad u_i < 0 \end{cases} \tag{41}$$

The functions $r(x)$, $g(x)$, and $h(x)$ in the preceding formulation are given by

$$r(x) = [I + \gamma(L_1 - K - K_1)]Q_1 \tag{42}$$

$$g(x) = 2[I + \gamma(L_1 - K - K_1)]Q_1' + \gamma(2L_3 + L_2 - K_2 + 2K'')Q_1 \tag{43}$$

$$h(x) = [I + \gamma(L_1 - K - K_1)]Q_1'' + \gamma(2L_3 + L_2 - K_2 + 2K')Q_1' + \gamma(L_4 + L_5 - K'')Q_1 \tag{44}$$

The quantities $I, K, K_1, L, L_1, L_2, L_3, L_4, L_5$, and Q_1 for the present problem are defined[11] as

$$I = 4bd^3/3, \qquad K = K' = K'' = 0 \tag{45}$$

$$L = (m - 1)I \exp\left(-2\alpha \frac{|x - x_c|}{d}\right) \tag{46}$$

$$L_1 = \left(1 - \frac{1}{m}\right) I \exp\left(-\alpha \frac{|x - x_c|}{d}\right) \exp\left(-2\beta \frac{|x - x_c|}{d}\right) \tag{47}$$

$$K_1 = \left(1 - \frac{1}{m}\right) I \exp\left(-2\beta \frac{|x - x_c|}{d}\right) \tag{48}$$

$$Q_1 = \frac{I + \gamma(L_1 - K - K_1)}{I + \gamma(L - 2K)} \tag{49}$$

and

$$L_2 + L_3 = L_1', \qquad L_4 + L_5 = L_3', \qquad K_2 = K_1' \tag{50}$$

where the constants α, β, and m determined in Ref. 22 are given by

$$\alpha = 1.276, \quad \beta = 21.94$$
$$m = \frac{1}{1 + (3/4)(a/d)^2 - (3/2)(a/d) - (1/8)(a/d)^3} \tag{51}$$

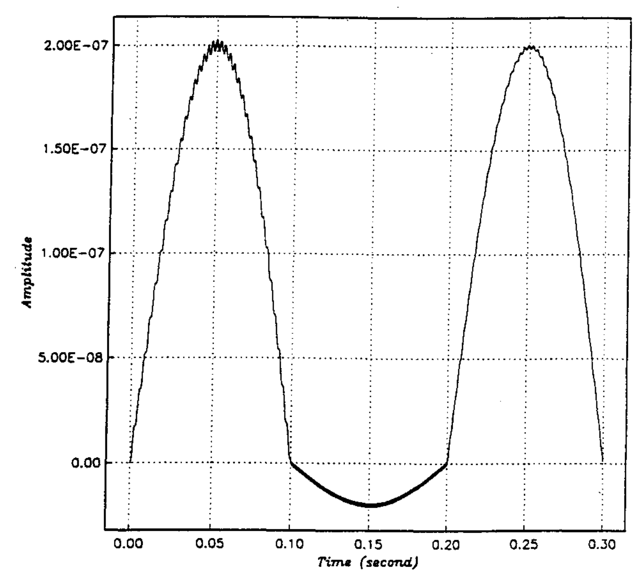

Fig. 5 Time history of a bilinear oscillator with $\Omega = 5$ Hz, $\omega_1 = 355.9$ Hz, and $\omega_2 = 1125.4$ Hz obtained from numerical simulation.

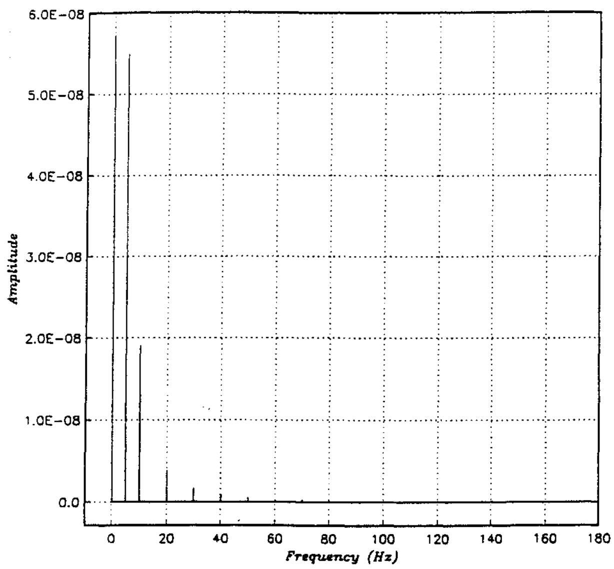

Fig. 6 Spectrum of the forced response of a bilinear oscillator with $\Omega = 5$ Hz, $\omega_1 = 355.9$ Hz, and $\omega_2 = 1125.4$ Hz obtained from numerical simulation.

To consider changes of the magnitude of excitation, the equation of motion for each vibrational mode of the cracked beam [Eq. (35)] is redefined as a bilinear oscillator with a bilinear forcing function:

$$\begin{cases} \ddot{u} + \omega_1^2 u = P_1 \sin \Omega t & \text{if} \quad u > 0 \\ \ddot{u} + \omega_2^2 u = P_2 \sin \Omega t & \text{if} \quad u < 0 \end{cases} \tag{52}$$

where P_1 is the magnitude of the external excitation when $u > 0$, and P_2 is the magnitude of the external excitation when $u < 0$. Their values for the ith mode are given by

$$P_1 = \frac{1}{m_0} \int_0^l \xi \psi_i^c \, dx, \qquad P_2 = \frac{1}{m_0} \int_0^l \xi \psi_i^{nc} \, dx \tag{53}$$

The particular solution for Eq. (53) is

$$\begin{cases} u_p(t) = [P_1/(\omega_1^2 - \Omega^2)] \sin \Omega t & \text{if} \quad u > 0 \\ u_p(t) = [P_2/(\omega_2^2 - \Omega^2)] \sin \Omega t & \text{if} \quad u < 0 \end{cases} \tag{54}$$

From the procedure described in the previous section and the homogeneous initial conditions, the solution of Eq. (52) is

$$u(t) = u_f(t) + u_s(t) \tag{55}$$

where the forced response u_f and the system resonance u_s are defined by

$$u_f(t) = \frac{1}{2}\left(\frac{P_1}{\omega_1^2 - \Omega^2} + \frac{P_2}{\omega_2^2 - \Omega^2}\right) \sin \Omega t + \left(\frac{P_1}{\omega_1^2 - \Omega^2} - \frac{P_2}{\omega_2^2 - \Omega^2}\right)\left\{\frac{1}{\pi} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2k\Omega t}{(2k+1)(2k-1)}\right\} \tag{56}$$

and

$$u_s(t) = ½[A_1 \sin \omega_1 t + A_2 \sin(\omega_2 t + \beta_0)] + \frac{A_1}{\pi} \sum_{k=1,3,5}^{\infty} \frac{\cos(\omega_1 - k\Omega)t - \cos(\omega_1 + k\Omega)t}{k} - \frac{A_2}{\pi} \sum_{k=1,3,5}^{\infty} \frac{\cos[(\omega_2 - k\Omega)t + \beta_0] - \cos[(\omega_2 + k\Omega)t + \beta_0]}{k} \tag{57}$$

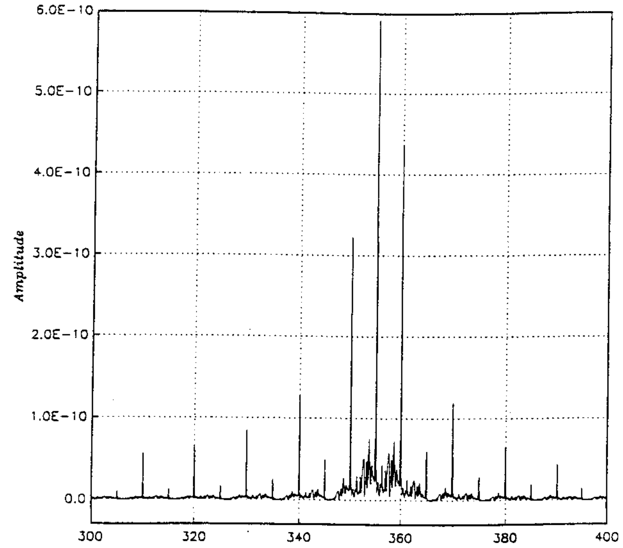

Fig. 7 Spectrum of the system resonance of a bilinear oscillator with $\Omega = 5$ Hz, $\omega_1 = 355.9$ Hz, and $\omega_2 = 1125.4$ Hz obtained from numerical simulation.

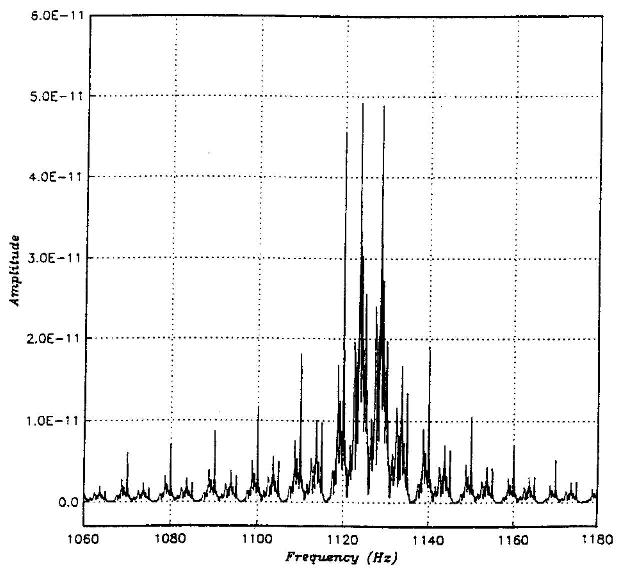

Fig. 8 Spectrum of the system resonance of a bilinear oscillator with $\Omega = 5$ Hz, $\omega_1 = 355.9$ Hz, and $\omega_2 = 1125.4$ Hz obtained from numerical simulation.

The constants A_1 and A_2 are given as

$$\begin{cases} A_1 = -\dfrac{\Omega}{\omega_1}\left(\dfrac{P_1}{\omega_1^2 - \Omega^2}\right) \\ A_2 = \dfrac{\Omega}{\omega_2}\left(\dfrac{P_2}{\omega_2^2 - \Omega^2}\right) \end{cases} \tag{58}$$

Equations (56) and (57) give the solution for forced vibrations for each vibrational mode of a cracked beam. It is clear that the preceding solution has the same spectral pattern as the solution in the previous section. The only difference is that the magnitude of each harmonic component has changed due to the bilinear forcing function.

Results and Discussion

To verify the solutions obtained in the previous sections, we have performed extensive numerical simulations using the procedure described in Ref. 11 and compared the numerical results with the prediction of the present analytical solution. The comparison is made for the cases of forced bilinear oscillators and forced vibrations of beams with a fatigue crack.

Forced Bilinear Oscillators

For bilinear oscillators under low-frequency excitation, we select the stiffness ratio $k_2/k_1 = 10$ to simulate a system with strong nonlinearity. The corresponding natural frequency ratio ω_1/ω_2, according to Eq. (2), is then $\sqrt{10}$. In this case the lowest system frequency ω_1 is selected to be 2236.07 rad/s (355.9 Hz), and ω_2 is then 7071.07 rad/s (1125.4 Hz). In accordance with the low-forcing-frequency assumption, we choose the forcing frequency to be $\Omega = 31.416$ rad/s (5 Hz) with magnitude $P = 1$, so that the forcing frequency is far from the lowest system frequency, ω_1.

The time history and the spectrum of the bilinear system were calculated using the same numerical scheme as that in Ref. 11, Figs. 5–8 show the time history and spectrum of the bilinear oscillator with the initial conditions $u(0) = 0.0$ and $\dot{u}(0) = 0.0$. From Figs. 6–8 one can see that the spectral pattern is the same as that predicted by Eqs. (32) and (33). Each component in Figs. 6–8 corresponds to a term in either the force response or the system resonance. Table 1 shows the frequency components in the spectra of the bilinear oscillator from both the numerical simulation and the present analysis. From these results one can see that, for the forced response, which has the components at 0, 1Ω, 2Ω, and 4Ω, the predictions of the present analysis agree well with the results from the simulation. However, as expected, a significant error is found at the system resonances ω_1 and ω_2. More specifically, the present closed-form solution gives an overestimate at ω_1 and an underestimate at ω_2. This is mainly due to the noise (which can easily be observed in Figs. 7 and 8) generated around the system resonance in the simulation. Another possible cause for the difference between the numerical results and the analytical predictions is that the chosen external excitation frequency, 5 Hz, is not low enough to satisfy the low-forcing-frequency assumption. Therefore, we expect that the error will be less for cases with higher system resonance frequencies and/or lower forcing frequency. A typical example is given for cracked beam dynamics in the following section.

Forced Vibrations of Cracked Beams

Consider the case of a simply supported beam with a fatigue crack of $a = d$, located at $x_c = l/2$, shown in Fig. 4. Under a concentrated external exciting force ($\Omega = 5$ Hz) at the midpoint of the beam, the frequencies and magnitudes of excitation for the first mode are determined to be $\omega_1 = 28.7$ Hz, $\omega_2 = 26.5$ Hz, $P_1 = 0.577$, and $P_2 = 0.552$. According to the closed-form solution [Eqs. (55–57)], the forced response of cracked beams should have components only at frequencies 0, Ω, and $2n\Omega$, which is exactly the same spectral pattern as that shown in Fig. 9b obtained from the simulation. For the system resonance shown in Fig. 9a there are several harmonics. The component with the largest magnitude occurs at the bilinear frequency ω_0. This component is actually the vector sum of the two components predicted to occur at ω_1 and ω_2 (the difference between ω_1 and ω_2 is so small that the components at these two frequencies cannot be discriminated by the FFT process). Other minor components are exactly $(2n - 1)\Omega$ away from the major component at ω_0, as predicted by the present solution.

Table 1 Components in the spectrum of a bilinear oscillator

Frequency, Hz	Results from present analysis	Results from simulation
0.0	5.73×10^{-8}	5.73×10^{-8}
$1\Omega = 5.0$	5.50×10^{-8}	5.50×10^{-8}
$2\Omega = 10.0$	1.91×10^{-8}	1.91×10^{-8}
$4\Omega = 20.0$	3.82×10^{-9}	3.83×10^{-9}
$\omega_1 = 355.9$	7.03×10^{-10}	5.91×10^{-10}
$\omega_2 = 1125.4$	2.22×10^{-11}	4.92×10^{-11}

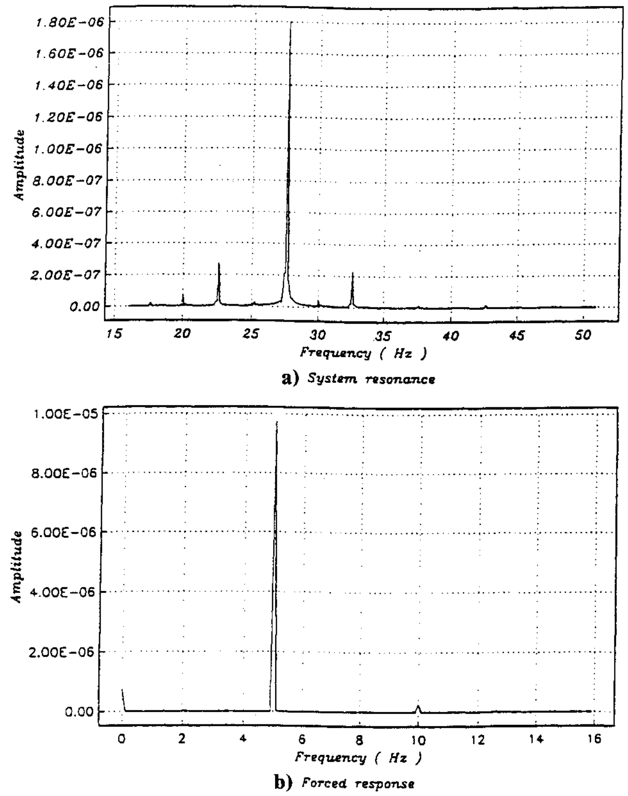

Fig. 9 Spectrum for the first mode of vibration of a simply supported cracked beam under the forcing frequency $\Omega = 5$ Hz obtained from numerical simulation.

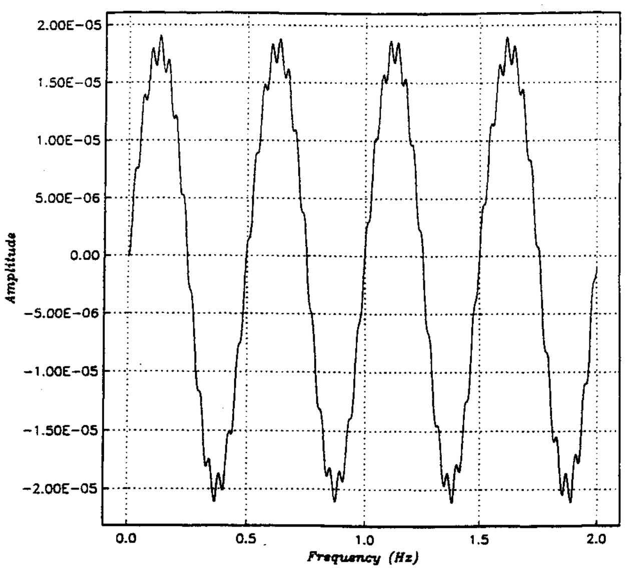

Fig. 10 Time history for the first mode of vibration of a simply supported cracked beam under the forcing frequency $\Omega = 2$ Hz obtained from numerical simulation.

Furthermore, the magnitude of each harmonic component in the spectrum predicted by the present theory is compared with the numerical results listed in Table 2. As one can see, the components of the forced response are well predicted by the present theory, except the one at 4Ω, which is very close to the major resonance at ω_0. A possible reason for the difference between the theoretical and numerical results is that the forcing frequency is not low enough compared with the bilinear frequency ω_0.

To see if the results could be improved by lowering the forcing frequency, we lower it from 5 Hz to 2 Hz and redo the simulation. Figures 10 and 11 show the time history and spectrum for the first mode vibration of a simply supported cracked beam under the forcing frequency $\Omega = 2$ Hz. Table 3 shows the comparison. As one can see, every component in the spectrum is well predicted by the theory, which means that, as long as the forcing frequency Ω is much lower than both natural frequencies ω_1 and ω_2, the present closed-form solution is capable of providing a satisfactory prediction for the spectrum of a cracked beam. It is also seen from the comparison listed in Tables 2 and 3 that this solution strongly depends on the assumption that the forcing frequency Ω be much smaller than both natural frequencies ω_1 and ω_2 of a cracked beam. A good prediction can only be obtained from the present solution if this assumption is valid.

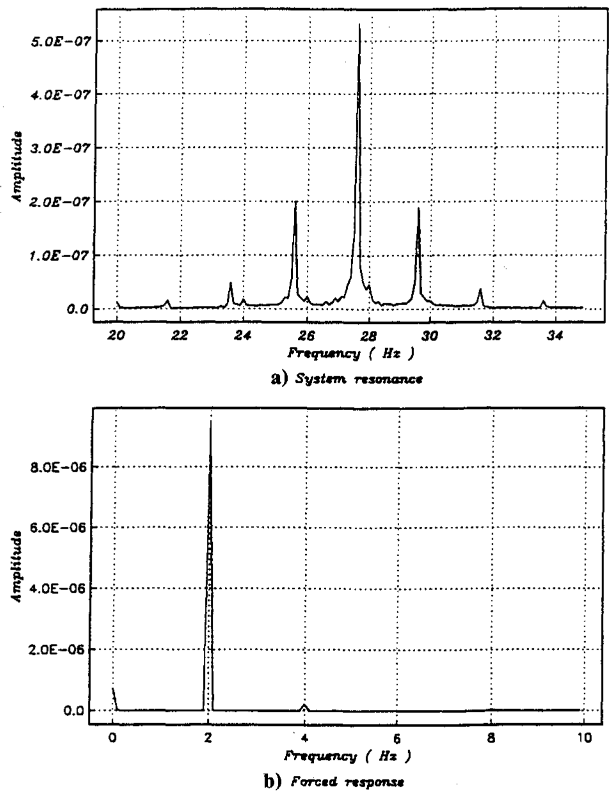

Fig. 11 Spectrum for the first mode of vibration of a simply supported cracked beam under the forcing frequency $\Omega = 2$ Hz obtained from numerical simulation.

Table 2 Components in the spectrum of a cracked beam with $\Omega = 5$ Hz

Frequency, Hz	Results from present analysis	Results from simulation
0.0	7.37×10^{-7}	7.25×10^{-7}
$1\Omega = 5.0$	9.75×10^{-6}	9.75×10^{-6}
$2\Omega = 10.0$	2.46×10^{-7}	2.60×10^{-7}
$4\Omega = 20.0$	4.91×10^{-8}	7.78×10^{-8}
$\omega_0 = 27.6$	1.64×10^{-6}	1.81×10^{-6}

Table 3 Components in the spectrum of a cracked beam with $\Omega = 2$ Hz

Frequency, Hz	Results from present analysis	Results from simulation
0.0	6.89×10^{-7}	6.87×10^{-7}
$1\Omega = 2.0$	9.47×10^{-6}	9.47×10^{-6}
$2\Omega = 4.0$	2.30×10^{-7}	2.31×10^{-7}
$4\Omega = 8.0$	4.59×10^{-8}	4.79×10^{-8}
$\omega_0 = 27.6$	5.39×10^{-7}	5.33×10^{-7}

The comparison between the numerical simulation and the analytical solution shows that both the spectral pattern and the magnitude of each harmonic component can be predicted well. In other words, spectral information from vibrations of cracked structures can provide a more precise way to identify the existence of structural damage, since most approaches for detecting crack existence depend on the frequency decrease due to the crack. As is well known, the natural frequencies of a structure may change due to not only structural damage, but also other factors, such as temperature. This weakens the uniqueness of the identification criterion. If a structural integrity monitoring system is built based on the open crack model, it will be hard to decide whether the frequency decrease is due to the existence of cracks or to environmental factors. A more effective way to identify the presence of structural damage can be developed via the present approach. By applying a low-frequency harmonic force to a structure, one can determine the existence of cracks from the spectral pattern of the structural vibrations.

Conclusion

In this study a closed-form solution for the forced vibration of a bilinear oscillator with constant-amplitude excitation, under the assumption that the forcing frequency is much lower than both natural frequencies of the system, has been obtained. This solution has been further modified for the case with a bilinear forcing function, which is the model for cracked beams derived from the Galerkin procedure. Both the spectral pattern and the magnitude of each harmonic component for a cracked beam predicted by the present analysis has been found to agree well with the results obtained from numerical simulation. This suggests a possible crack detecting approach using low-frequency harmonic excitation. Since the detection criterion is based on the bilinear behavior shown in the spectrum, which will not be affected by environmental factors, the proposed approach may improve the accuracy and reliability of damage identification procedures.

Further investigation of the solutions for damped forced vibrations and experimental verification of the present analysis will be done by the authors. A better understanding of the dynamic behavior of cracked structures may be achieved by the study of vibrations of bilinear oscillators.

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